RATE MEASURE

Velocity, Acceleration:

A particle starts from a fixed point and moves a distance 'S' along a straight-line during time 't' then

$s = f(t)$

Velocity $= v = \frac{ds}{dt}$

Acceleration $= a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Note:

(i) if $v > 0$, then the particle moving away from the straight point.

(ii) If $v < 0$, then particle moving towards the straight point.

(iii) If $v = 0$, then the particle comes rest.

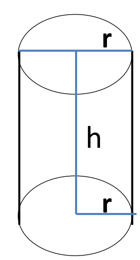
The Following formulae will be used in Solving problems

Rectangle:

Perimeter of Rectangle $= 2\,(l + b)$ units

Area of Rectangle $= l \times b$ square units

CYLINDER:



If 'r is the radius of the base of cylinder and 'h' is the height of the cylinder, then

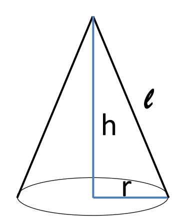
Area of base $= \pi r^2$ sq.u.

Curved surface area $= 2\pi rh$ sq.u.

Total surface area $= 2\pi r\,(h + r)$ sq.u.

Volume $= \pi r^2 h$ cubic units.

CONE:



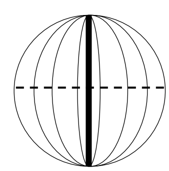
If 'r' is the radius of base, 'h' is the height of cone and 'ℓ' is slant height then

$\ell^2 = r^2 + h^2$

Curved surface area $= \pi r\ell$ units.

Total surface area $= \pi r\,(\ell + r)$ sq. units. Volume $= \frac{1}{3}\pi r^2 h$ cubic units.

SPHERE:



If 'r' is the radius of the Sphere then

Surface area $= \pi r^2$ sq. units.

Volume $= \frac{4}{3}\pi r^3$ cubic units.

Solved Problems:

1. A particle is moving along a line such that $S = t^2 - 6t + 8$. Find its velocity and acceleration at $t = 2$ sec.

   Sol: Given $S = t^2 - 6t + 8$

   $$\text{Velocity} = \frac{dS}{dt} = \frac{d}{dt}(t^2 - 6t + 8) = 2t - 6$$

   at $t = 2$ sec, Velocity $= 2(2) - 6 = 4 - 6 = -2$ units/sec

   $$\text{Acceleration (a)} = \frac{dv}{dt} = 2 - 0 = 2$$

   at $t = 2$ sec, Acceleration $= 2$ units/sec^2

2. A particle is moving along a straight line according to the law $S = 2t^3 - 3t^2 + 15t + 18$. Find its velocity when its acceleration is zero

   Sol: Given $S = 2t^3 - 3t^2 + 15t + 18$

   $$\text{Velocity} = \frac{dS}{dt} = \frac{d}{dt}(2t^3 - 3t^2 + 15t + 18) = 6t^2 - 6t + 15$$

   $$\text{Acceleration (a)} = \frac{dv}{dt} = \frac{d}{dt}(6t^2 - 6t + 15) = 12t - 6$$

   When the acceleration is zero

   $\Longrightarrow 12t - 6 = 0$

   $\Longrightarrow 12t = 6$

   $\Longrightarrow t = \frac{6}{12} = \frac{1}{2}$

   When $t = \frac{1}{2}$ sec, velocity $= 6\left(\frac{1}{2}\right)^2 - 6\left(\frac{1}{2}\right) + 15$

   $= \frac{6}{4} - 3 + 15$

   $= \frac{3}{2} + 12 = \frac{3+24}{2} = \frac{27}{2}$ m/sec

3. A stone is projected vertically up ward at a height S foot after 1 second, where $S = 80t - 16t^2$. Find (i) Its initial velocity (ii) Its velocity at $t = 1$ sec (iii) Time when it is at rest (iv) The greatest height above the ground

   Sol: Given $S = 80t - 16t^2$

   $$\text{Velocity} = \frac{dS}{dt} = \frac{d}{dt}(80t - 16t^2) = 80 - 16(2t) = 80 - 32t$$

   (i) Initial velocity

   When $t = 0$ sec, we get initial velocity

   $\Longrightarrow$ Initial velocity $= 80 - 32(0) = 80$ ft/sec

   $\therefore$ Initial velocity 80 ft/sec

   (ii) Velocity at $t = 1$ sec

   At $t = 1$ sec, Velocity $= 80 - 32(1) = 80 - 32 = 48$ ft/sec

   $\therefore$ at $t = 1$ sec, the velocity $= 48$ ft/sec

   (iii) Time when stone is at rest

When the stone is rest, then its velocity is zero

$\Longrightarrow 80 - 32t = 0$

$80 = 32t$

$t = \frac{80}{32} = \frac{5}{2}$ sec

$\therefore$ time $= \frac{5}{2}$ sec

(iv) The greatest height above the ground

When the stone is greatest height above the ground, then its velocity is zero

$80 - 32t = 0$

$80 = 32t$

$t = \frac{80}{32} = \frac{5}{2}$ sec

$S = 80t - 16t^2$

$= 80\left(\frac{5}{2}\right) - 16\left(\frac{5}{2}\right)^2$

$= 200 - 16\times\frac{25}{4} = 200 - 100 = 100$

$\therefore$ The greatest height above the ground = 100 ft

4. A circular metal plate expands by heat so that its radius is increasing at the rate of 0.02cm/sec. At what rate its area increasing when the radius is 20cm?

Sol: Let radius of the circle = r and area of circle = A

r= 20 cm and $\frac{dr}{dt} = 0.02$ m/sec

$A = \pi r^2$

$\frac{dA}{dt} = \pi\, 2r\frac{dr}{dt}$

$= 2\pi \times 20 \times 0.02$

$= \pi \times 40 \times 0.02$

$= 0.8\,\pi\ cm^2/sec$

$\therefore$ rate of change in area of circle is $0.8\,\pi\ cm^2/sec$

5. The radius of a circle is increasing at the rate of 2 cm/sec. Find the rate of change of the area when the radius is 24 cm.

Sol: Let radius of the circle = r and area of circle = A

r= 24 cm and $\frac{dr}{dt} = 2$ m/sec

$A = \pi r^2$

$\frac{dA}{dt} = \pi\, 2r\frac{dr}{dt}$

$= 2\pi \times 24 \times 2$

$= \pi \times 48 \times 2$

$= 96\ \pi\ cm^2/sec$

$\therefore$ rate of change in area of circle is $96\ \pi\ cm^2/sec$

6. A circular metal plate expands by heat so that its radius is increasing at the rate of 0.01cm/sec. At what rate its area increasing when the radius is 2cm?

Sol: Let radius of the circle = r and area of circle = A

$r = 2$ cm and $\frac{dr}{dt} = 0.01$ m/sec

$A = \pi r^2$

$\frac{dA}{dt} = \pi\, 2r \frac{dr}{dt}$

$= 2\pi \times 2 \times 0.01$

$= \pi \times 4 \times 0.01$

$= 0.04\pi\ cm^2/sec$

$\therefore$ rate of change in area of circle is $0.04\ \pi\ cm^2/sec$

7. Each side of a square increases at the rate of 1.5 cm/sec. Find the rate at which the area of the square when the side is 12 cm. Also find the rate at which perimeter increases.

Sol: Let the side of square = x, Area of square = A and perimeter of square = P

$x = 12$ cm and $\frac{dx}{dt} = 1.5$ cm/sc

$A = x^2$

$\frac{dA}{dt} = 2x \frac{dx}{dt}$

$= 2 \times 12 \times 1.5$

$= 36\ cm^2/sec$

$P = 4x$

$\frac{dP}{dt} = 4 \frac{dx}{dt}$

$= 4 \times 1.5$

$= 6$ cm/sec

8. Each side of a square increases at the rate of 1 cm/sec. Find the rate at which the area of the square when the side is 15 cm. Also find the rate at which perimeter increases.

Sol: Let the side of square = x, Area of square = A and perimeter of square = P

$x = 15$ cm and $\frac{dx}{dt} = 1$ cm/sc

$A = x^2$

$\frac{dA}{dt} = 2x \frac{dx}{dt}$

$= 2 \times 15 \times 1$

$= 30\ cm^2/sec$

$P = 4x$

$\frac{dP}{dt} = 4 \frac{dx}{dt}$

$= 4 \times 1$

$= 4 \text{ cm/sec}$

9. Each side of a square increases at the rate of 0.3 cm/sec. Find the rate at which the area of the square when the side is 8 cm. Also find the rate at which perimeter increases.

Sol: Let the side of square = x, Area of square = A and perimeter of square = P

$x = 8 \text{ cm and } \frac{dx}{dt} = 0.3 \text{ cm/sc}$

$A = x^2$

$\frac{dA}{dt} = 2x\frac{dx}{dt}$

$= 2 \times 8 \times 0.3$

$= 4.8 \text{ cm}^2\text{/sec}$

$P = 4x$

$\frac{dP}{dt} = 4\frac{dx}{dt}$

$= 4 \times 0.3$

$= 1.2 \text{ cm/sec}$

10. A spherical balloon is being inflated so that the radius is increasing at the rate of 3 cm/sec. Find the rate at which the volume is increasing when r = 10 cm.

Sol: : Let the radius of spherical balloon = r , volume of spherical balloon = V

$r = 3 \text{ cm and } \frac{dr}{dt} = 3 \text{ cm/sc}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2\frac{dr}{dt}$

$= 4\pi \times 3(10)^2 = 12 \times 100\,\pi$

$= 1200\pi \text{ cm}^3\text{/sec}$

11. The Volume of sphere is increasing at the rate of 400cm³/sec. Find the rate of increasing of its radius and its surface area at the instant when the radius of sphere is 40 cm.

Sol: Let the radius of sphere = r, volume = V and surface area = S

$r = 40 \text{ cm and } \frac{dv}{dt} = 400 \text{ cm}^3\text{/sc}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2\frac{dr}{dt}$

$400 = 4\pi \times (40)^2\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{400}{6400\pi} = \frac{1}{16\pi} \text{ cm/sec}$

$S = 4\pi r^2$

$\frac{dS}{dt} = 4\,\pi \times 2r\frac{dr}{dt}$

$= 4\,\pi \times 2\,(40)\,\frac{1}{16\pi}$

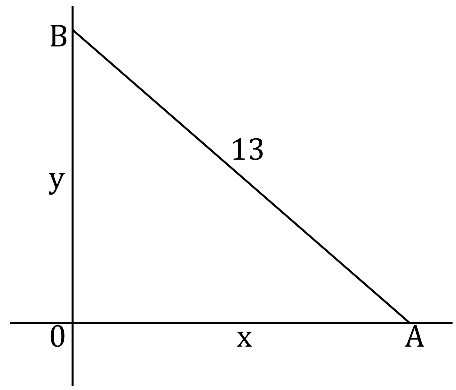
$= 20 \text{ cm}^2\text{/sec}$

12. A ladder 13 meters long leans against a vertical wall. If the foot of the ladder is pulled away from the wall at the rate of 1 m/sec along the horizontal floor, how fast is the top descending, when the lower end is 12 m away from the wall.

Sol: Let OA = x meters

OB = y meters

$x = 12$ cm and $\frac{dx}{dt} = 1\text{m/sec}$



in ΔAOB $AB^2 = OA^2 + OB^2$

$13^2 = x^2 + y^2$

$169 = x^2 + y^2$

$y^2 = 169 - x^2$

$y^2 = 169 - 12^2$

$= 169 - 144 = 25$

$y = 5$

$x^2 + y^2 = 169$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2x\,(1) + 2y\frac{dy}{dt} = 0$

$2y\frac{dy}{dt} = -2x$

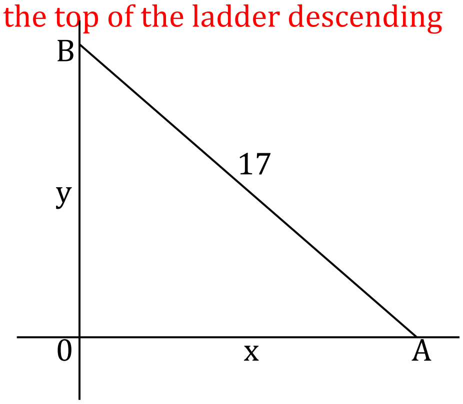
$\frac{dy}{dt} = \frac{-2x}{2y} = \frac{-x}{y} = \frac{-12}{5}$ m/sec

13. One end of a ladder 17 ft long is leaning against a wall. If the foot of the ladder be pulled away from the wall at the rate of 3ft/min, how fast is the top of the ladder descending when the foot of the ladder is 8 ft from the wall?

Sol: Let OA = x ft

OB = y ft

$x = 8$ ft and $\frac{dx}{dt} = 3$ ft/min



in ΔAOB $AB^2 = OA^2 + OB^2$

$17^2 = x^2 + y^2$

$289 = x^2 + y^2$

$y^2 = 289 - x^2$

$y^2 = 289 - 8^2$

$= 289 - 64 = 225$

$y = 15$

$x^2 + y^2 = 289$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2x\,(3) + 2y\frac{dy}{dt} = 0$

$2y\frac{dy}{dt} = -6x$

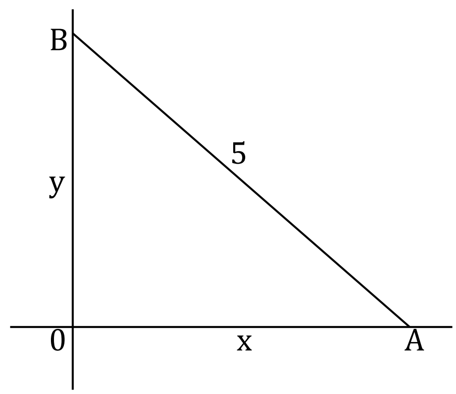
$\frac{dy}{dt} = \frac{-6x}{2y} = \frac{-3x}{y} = \frac{-3\times8}{15} = \frac{-8}{5} = -1.6$ ft/min

14. A ladder 5m long is placed against a vertical wall. Foot of the ladder is slipping away from the wall at the rate of 5 cm/sec. Find the rate descending of its top if the foot of the ladder is 3m away from the wall.

Sol: Let OA = x meters

OB = y meters

$x = 3$ m $= 300$cm and $\frac{dx}{dt} = 5$cm/sec



in ΔAOB $AB^2 = OA^2 + OB^2$

$5^2 = x^2 + y^2$

$25 = x^2 + y^2$

$y^2 = 25 - x^2$

$y^2 = 25 - 3^2$

$= 25 - 9 = 16$

$y = 4m = 400cm$

$x^2 + y^2 = 25$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$2x(5) + 2y\frac{dy}{dt} = 0$

$2y\frac{dy}{dt} = -10x$

$\frac{dy}{dt} = \frac{-10x}{2y} = \frac{-5x}{y} = \frac{-5\times300}{400} = \frac{-15}{4}$ cm/sec

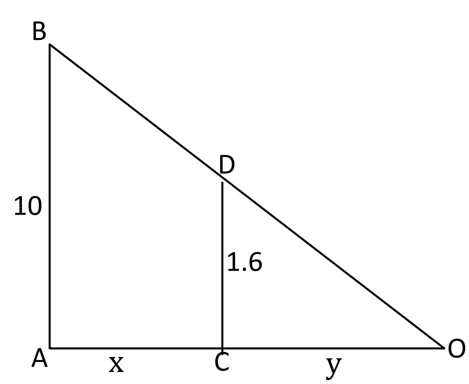
15. A boy 1.6 m tall is walking away from a lamp post 10m tall. If the boy is walking at the rate of 1.2m/sec, how fast is his shadow increasing when he is 15 m from the lamp post?

Sol: Let AB = height of the lamp post = 10 cm

CD = height of the boy = 1.6 m

AC = x and CO = y

x= 15 cm and $\frac{dx}{dt} = 1.2$ cm/sec



ΔOAB ~ ΔOCD

$\frac{AB}{OA} = \frac{CD}{OC} \Rightarrow \frac{10}{x+y} = \frac{1.6}{y}$

$\Rightarrow 10y = 1.6(x+y)$

$\Rightarrow 10y = 1.6x + 1.6y$

$\Rightarrow 10y - 1.6y = 1.6x \Rightarrow 8.4y = 1.6x$

$\Rightarrow y = \frac{1.6x}{8.4}$

$$\frac{dy}{dt} = \frac{1.6}{8.4}\frac{dx}{dt}$$
$$= \frac{1.6}{8.4} \times 1.2$$
$$= 0.228 \text{ m/sec}$$

∴ The shadow is increasing at the rate of 0.228 m/sec

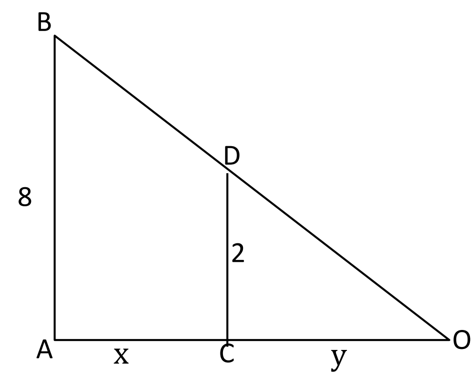
16. A man of 2m tall is approaching a lamp post at the rate of 0.5 m/sec. If the lamp post is situated at a height of 8 m, then find the rate at which the length of the shadow decreasing?

Sol: Let AB = height of the lamp post = 8 cm

CD = height of the man = 2 m

AC = x and CO = y

$\frac{dx}{dt} = 0.5$ m/sec



$\Delta OAB \sim \Delta OCD$

$\frac{AB}{OA} = \frac{CD}{OC} \Longrightarrow \frac{8}{x+y} = \frac{2}{y}$

$\Longrightarrow 8y = 2\,(x+ y)$

$\Longrightarrow 4y = x + y$

$\Longrightarrow 4y - y = x \Longrightarrow 3y = x$

$\Longrightarrow y = \frac{x}{3}$

$$\frac{dy}{dt} = \frac{1}{3}\frac{dx}{dt}$$
$$= \frac{1}{3} \times 0.5$$
$$= 0.166 \text{ m/sec}$$

∴ The shadow is increasing at the rate of 0.228 m/sec

17. The base radius of cylindrical vessel full of oil is 30cm. Oil is drawn at the rate of 27000 cm^3/min. Find the rate at which the level of the oil is falling in the vessel.

Sol: Let radius of the cylinder = r, height = h and Volume = V

r= 30 cm and $\frac{dV}{dt} = -\,27000\ cm^3/min$

Volume of the cylinder = $\pi r^2 h$

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$
$$-\,27000 = \pi r^2 \frac{dh}{dt}$$
$$\Longrightarrow \frac{dh}{dt} = \frac{1}{\pi r^2}(-\,27000)$$
$$= \frac{1}{\pi (30)^2}(-\,27000)$$
$$= \frac{-30}{\pi} \text{ cm/min}$$

∴ The rate at which the level of oil is falling $= \frac{-30}{\pi}$ cm/min

18. A cylindrical vessel of radius 0.5 m is being filled with the oil at the rate of $\frac{0.25}{10^6}$ m^3/min. How fast is the oil in the vessel rises?

Sol: Let radius of the cylinder = r, height = h and Volume = V

$r = 0.5$ m and $\frac{dV}{dt} = \frac{0.25}{10^6} m^3/min$

Volume of the cylinder $= \pi r^2 h$

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$$

$$\frac{0.25}{10^6} = \pi r^2 \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{\pi r^2} \frac{0.25}{10^6}$$

$$= \frac{1}{\pi (0.5)^2} \frac{0.25}{10^6}$$

$$= \frac{1}{\pi . 10^6}$$

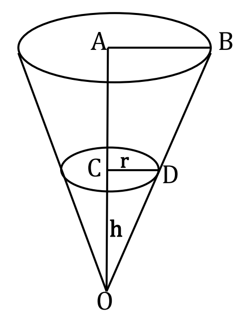
$\therefore$ The rate at which the level of oil is raising $= \frac{1}{10^6 \pi}$ cm/min

19. A conical water tank with vertex down has 20m and depth of 20 m. Water is being pumped into the tank at the rate of $40 m^3/min$. How fast is the level of the water rising when the water is 8 m depth?

Sol: Let radius of the Cone = r, height = h and Volume = V

$r = AB = 20$ m, height of the cone $= OA = 20$ m and $\frac{dV}{dt} = 40 m^3/min$

let $CD = r$ and $OC = h$



$\Delta OAB \sim \Delta OCD$

$$\frac{AB}{OA} = \frac{CD}{OC} \Rightarrow \frac{20}{20} = \frac{r}{h}$$

$$\Rightarrow r = h$$

Volume of the Cone is

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi (h)^2 h = \frac{1}{3}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{3}\pi 3h^2 \frac{dh}{dt}$$

$$40 = \pi h^2 \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{\pi h^2} 40$$

$$= \frac{1}{\pi (8)^2} 40$$

$$= \frac{5}{8\pi} m/min$$

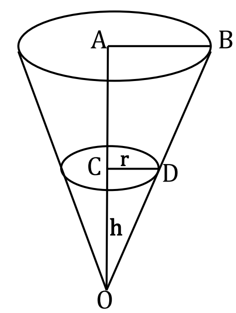
$\therefore$ Rate of change in water level $= \frac{5}{8\pi}$ m/min

20. Water is poured into a conical flask of height 10cm and of diameter of base 4cm at the rate of 0.3 cm^3/sec. Find the rate at which the level of water is increasing when the depth of water is 3 cm

Sol: Let radius of the Cone = r, height = h and Volume = V

$r = AB = 2c\ m$, height of the cone $= OA = 10$ cm and $\frac{dV}{dt} = 0.3\ cm^3/min$

let CD = r and OC = h



$\Delta OAB \sim \Delta OCD$

$$\frac{AB}{OA} = \frac{CD}{OC} \Longrightarrow \frac{2}{10} = \frac{r}{h}$$

$$\Longrightarrow 10r = 2h$$

$$\Longrightarrow 5r = 2h$$

$$r = \frac{h}{5}$$

Volume of the Cone is

$$V = \frac{1}{3}\pi\, r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{h}{5}\right)^2 h \quad = \frac{1}{3}\pi \frac{1}{25}\ h^3$$

$$\frac{dV}{dt} = \frac{1}{3}\pi \frac{1}{25}\ 3h^2 \frac{dh}{dt}$$

$$0.3 = \frac{1}{25}\ \pi\, h^2 \frac{dh}{dt}$$

$$\Longrightarrow \frac{dh}{dt} = \frac{25}{\pi h^2}\ (0.3)$$

$$= \frac{25}{\pi(3)^2}(0.3) = \frac{25}{\pi\,.9}(0.3) = \frac{25}{\pi\,.9}\ \frac{3}{10}$$

$$= \frac{5}{6\pi}\ m/min$$

$\therefore$ Rate of change in water level $= \frac{5}{6\pi}\ m/min$